

A GEOMETRIC JACQUET FUNCTOR

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1. INTRODUCTION

In the paper [BB1], Beilinson and Bernstein used the method of localisation to give a new proof and generalisation of Casselman's subrepresentation theorem. The key point is to interpret \mathfrak{n} -homology in geometric terms. The object of this note is to go one step further and describe the Jacquet module functor on Harish-Chandra modules via geometry.

Let $G_{\mathbb{R}}$ be a real reductive linear algebraic group, and let $K_{\mathbb{R}}$ be a maximal compact subgroup of $G_{\mathbb{R}}$. We use lower-case gothic letters to denote the corresponding Lie algebras, and omit the subscript “ \mathbb{R} ” to denote complexifications. Thus (\mathfrak{g}, K) denotes the Harish-Chandra pair corresponding to $G_{\mathbb{R}}$.

Let \mathfrak{h} be the universal Cartan of \mathfrak{g} , that is $\mathfrak{h} = \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ where \mathfrak{b} is any Borel of \mathfrak{g} . We equip \mathfrak{h} with the usual choice of positive roots by declaring the roots of \mathfrak{b} to be negative. We write $\rho \in \mathfrak{h}^*$ for half the sum of the positive roots. To any $\lambda \in \mathfrak{h}^*$ we associate a character χ_{λ} of the centre $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ via the Harish-Chandra homomorphism. Under this correspondence, the element $\rho \in \mathfrak{h}^*$ corresponds to the trivial character χ_{ρ} . For the rest of this paper, we work with $\lambda \in \mathfrak{h}^*$ that is dominant, i.e. $\check{\alpha}(\lambda) \notin \{-1, -2, \dots\}$ for any positive coroot $\check{\alpha}$.

A Harish-Chandra module with infinitesimal character λ is, by definition, a (\mathfrak{g}, K) -module which is finitely generated over $U(\mathfrak{g})$ and on which $Z(\mathfrak{g})$ acts via the character χ_{λ} . We can also view Harish-Chandra modules with infinitesimal character λ as modules over the ring U_{λ} which is the quotient of $U(\mathfrak{g})$ by the two-sided ideal generated by $\{z - \chi_{\lambda}(z) \mid z \in Z(\mathfrak{g})\}$. In light of this, we will sometimes refer to such Harish-Chandra modules simply as (U_{λ}, K) -modules.

Let X be the flag manifold of \mathfrak{g} , and let \mathcal{D}_{λ} be the sheaf of twisted differential operators with twist λ . By a $(\mathcal{D}_{\lambda}, K)$ -module we mean a coherent \mathcal{D}_{λ} -module which is K -equivariant. Such a \mathcal{D}_{λ} -module is, by necessity, regular holonomic since K acts on X with finitely many orbits. According to Beilinson-Bernstein [BB2], we have $\Gamma(X, \mathcal{D}_{\lambda}) = U_{\lambda}$, and the global sections functor

$$(1.1) \quad \Gamma : \{(\mathcal{D}_{\lambda}, K)\text{-modules}\} \longrightarrow \{(U_{\lambda}, K)\text{-modules}\}$$

is exact and essentially surjective. A section of the functor Γ is given by the localisation functor, which takes a (U_{λ}, K) -module M to the $(\mathcal{D}_{\lambda}, K)$ -module $\mathcal{D}_{\lambda} \otimes_{U_{\lambda}} M$. The localisation functor is an equivalence if λ is regular.

The authors would like to acknowledge the partial support of NSF grants DMS-0241562 and DMS-0401545 (M.E.), an NSF postdoctoral fellowship (D.N.), and NSF grant DMS-0105256 and DARPA grants MDA972-03-1-0030 and HR0011-04-1-0031 (K.V.).

Let $P_{\mathbb{R}}$ be a minimal parabolic subgroup of $G_{\mathbb{R}}$, let $\mathfrak{n}_{\mathbb{R}}$ be the nilpotent radical of $\mathfrak{p}_{\mathbb{R}}$, and let $\bar{\mathfrak{n}}_{\mathbb{R}}$ be the nilpotent subalgebra of $\mathfrak{g}_{\mathbb{R}}$ opposite to $\mathfrak{n}_{\mathbb{R}}$. If $A_{\mathbb{R}}$ denotes the maximal split subtorus of the centre of a Levi factor of $P_{\mathbb{R}}$, then $G_{\mathbb{R}}$ admits the Iwasawa decomposition $G_{\mathbb{R}} = K_{\mathbb{R}}A_{\mathbb{R}}N_{\mathbb{R}}$.

Given a (U_{λ}, K) -module M , we define the Jacquet module of M by the formula

$$(1.2) \quad J(M) = \mathfrak{n}\text{-finite vectors in } \hat{M},$$

where \hat{M} is the $\bar{\mathfrak{n}}$ -adic completion $\varprojlim M/\bar{\mathfrak{n}}^k M$ of M . We call the functor J which takes M to $J(M)$ the Jacquet functor. As we will explain in the next section, this functor, which is covariant and preserves the infinitesimal character, is dual to the usual Jacquet functor. The module $J(M)$ is a (U_{λ}, N) -module (in the sense that it is a Harish-Chandra module for the pair (\mathfrak{g}, N) with infinitesimal character λ).

To define the geometric Jacquet functor, we proceed as follows. Let $\nu : \mathbb{G}_m \rightarrow A$ be a cocharacter of G which is positive on the roots in \mathfrak{n} . By composing ν with the left action of G on X , we obtain an action of \mathbb{G}_m on X with action map

$$(1.3) \quad a : \mathbb{G}_m \times X \rightarrow X.$$

Consider the diagram

$$(1.4) \quad \mathbb{G}_m \times X \xrightarrow{j} \mathbb{A}^1 \times X \xleftarrow{i} \{0\} \times X \xrightarrow{\sim} X$$

where the maps j and i are the obvious inclusions. To a $(\mathcal{D}_{\lambda}, K)$ -module \mathcal{M} , we associate a \mathcal{D}_{λ} -module $\Psi(\mathcal{M})$ by taking the nearby cycles of $j_* a^* \mathcal{M}$ along $\{0\} \times X \xrightarrow{\sim} X$.

Before stating our main theorem, let us note that the localisation theory discussed above for (U_{λ}, K) -modules applies equally well to (U_{λ}, N) -modules.

1.1. Theorem. *(i) The localisation functor $M \mapsto \mathcal{D}_{\lambda} \otimes_{U_{\lambda}} M$ takes the Jacquet functor J to the geometric Jacquet functor Ψ , i.e.,*

$$\mathcal{D}_{\lambda} \otimes_{U_{\lambda}} J(M) = \Psi(\mathcal{D}_{\lambda} \otimes_{U_{\lambda}} M).$$

(ii) The global sections functor Γ takes the geometric Jacquet functor Ψ to the Jacquet functor J , i.e.,

$$\Gamma(\Psi(\mathcal{M})) = J(\Gamma(\mathcal{M})).$$

1.2. Remark. While the proof that we give of the main theorem is entirely algebraic, its motivation is quite geometric. Section 5 below provides a discussion of its geometric interpretation: via the deRham functor DR the category of $(\mathcal{D}_{\lambda}, K)$ -modules is equivalent to the category of λ -twisted K -equivariant perverse sheaves on X , and via this equivalence, Ψ may be viewed as the operation of nearby cycles on these perverse sheaves. We also discuss explicitly the case of $G_{\mathbb{R}} = \mathrm{SL}_2(\mathbb{R})$.

1.3. Remark. The functor Ψ takes $(\mathcal{D}_{\lambda}, K)$ -modules to $(\mathcal{D}_{\lambda}, N)$ -modules. This follows immediately from the above theorem and the fact that by the translation principle we may assume that λ is regular.

1.4. Remark. In this paper we could replace the ‘‘Iwasawa nilpotent subalgebra’’ \mathfrak{n} by any larger nilpotent subalgebra.

1.5. Remark. The problem of giving a geometric construction of the Jacquet functor was also considered by Casian and Collingwood. In their paper [CC], they describe a nearby cycles construction on Harish-Chandra sheaves (with dominant integral infinitesimal character); they refer to the induced functor on K -groups (taking into account weight filtrations) as a mixed Jacquet functor. They conjecture that this functor induces the usual Jacquet functor on Harish-Chandra modules.

The nearby cycles construction of [CC] coincides with the construction appearing in the definition of the functor Ψ . (Loosely, they pass to the limit with respect to one simple coroot at a time, while in the construction of Ψ , we pass to the limit with respect to all simple coroots at once.) Thus, Theorem 1.1 establishes the conjecture of Casian and Collingwood.

2. JACQUET FUNCTORS

In this section we recall some standard facts about the Jacquet functor as it is usually defined, and discuss the relation between this functor and the functor J defined in the Introduction. As a general reference for basic facts, we use [W], Chapter 4.

If M is a U_λ -module, then the Jacquet module of M with respect to $\bar{\mathfrak{n}}$ is defined by the formula

$$(2.1) \quad \check{J}(M) = \bar{\mathfrak{n}}\text{-finite vectors in } M^*,$$

where M^* is the algebraic dual of M . Of course, the infinitesimal character of $\check{J}(M)$ is $-\chi_\lambda = \chi_{\lambda^*}$, where $\lambda^* = -\lambda + 2\rho$.

2.1. Lemma. *The functor \check{J} is an exact contravariant functor from (U_λ, K) -modules (or more generally, U_λ -modules that are finitely generated over $U(\bar{\mathfrak{n}})$) to (U_{λ^*}, \bar{N}) -modules.*

Proof. The Lemma of Osborne guarantees that any (U_λ, K) -module is finitely generated over $U(\bar{\mathfrak{n}})$. If M is such a finitely generated module, then $\check{J}(M)$ is finitely generated over $U(\mathfrak{g})$, and for any positive integer k , the space of $\bar{\mathfrak{n}}^k$ -torsion vectors in $\check{J}(M)$ is finite-dimensional. (It is dual to the finite-dimensional space $M/\bar{\mathfrak{n}}^k M$.) Thus $\check{J}(M)$ is a Harish-Chandra module for (\mathfrak{g}, \bar{N}) .

One may regard $\check{J}(M)$ as the topological dual of the $\bar{\mathfrak{n}}$ -adic completion \hat{M} of M (equipped with its $\bar{\mathfrak{n}}$ -adic topology). The exactness of \check{J} is thus implied by the exactness of forming $\bar{\mathfrak{n}}$ -adic completions. This in turn follows from the fact that since $\bar{\mathfrak{n}}$ is nilpotent the usual Artin-Rees Lemma holds for the non-commutative ring $U(\bar{\mathfrak{n}})$. \square

Recall our definition of the Jacquet module $J(M)$ as \mathfrak{n} -finite vectors in the $\bar{\mathfrak{n}}$ -adic completion \hat{M} of M .

2.2. Lemma. *The functor J is an exact covariant functor from (U_λ, K) -modules to (U_λ, N) -modules.*

Proof. First note that one may view \hat{M} as the algebraic dual $\check{J}(M)^*$ of $\check{J}(M)$. The decomposition $\mathfrak{g} = \mathfrak{n} \oplus \bar{\mathfrak{p}}$ allows us to write $U(\mathfrak{g}) = U(\mathfrak{n}) \otimes_{\mathbb{C}} U(\bar{\mathfrak{p}})$ as \mathbb{C} -vector spaces. If M is a (U_λ, K) -module then for a sufficiently large choice of k , $\check{J}(M)$ is generated over $U(\mathfrak{g})$ by the space $\check{M}[\bar{\mathfrak{n}}^k]$ of $\bar{\mathfrak{n}}^k$ -torsion elements in \check{M} . Since $U(\bar{\mathfrak{p}})$ leaves $\check{M}[\bar{\mathfrak{n}}^k]$

invariant, we see that $\check{J}(M)$ is finitely generated over $U(\mathfrak{n})$. The lemma is now a consequence of the preceding one (with \mathfrak{n} replaced by $\bar{\mathfrak{n}}$). \square

We will now give a more concrete description of $J(M)$ which will be useful for our purposes. Recall from the last section that we have a cocharacter $\nu : \mathbb{G}_m \rightarrow G$ which when paired with the roots of \mathfrak{g} is positive precisely on the roots in \mathfrak{n} . Let us write h for the semisimple element in \mathfrak{g} given by $d\nu(t\partial_t)$. (For any coordinate t on \mathbb{G}_m , the Euler vector field $t\partial_t$ gives an element of the Lie algebra of \mathbb{G}_m independent of the choice of coordinate.) Note that the centraliser of h is precisely the Levi factor \mathfrak{l} of \mathfrak{p} , and the weights of h acting on \mathfrak{g} are integral.

For each natural number k , the quotient $M/\bar{\mathfrak{n}}^k M$ is a finite-dimensional vector space on which h acts, and so may be written as a direct sum of generalised h -eigenspaces. The surjection $M/\bar{\mathfrak{n}}^{k+1} M \rightarrow M/\bar{\mathfrak{n}}^k M$ is h -equivariant, and so induces surjections of the corresponding generalised h -eigenspaces.

As usual, we write \hat{M} for the $\bar{\mathfrak{n}}$ -adic completion $\varprojlim M/\bar{\mathfrak{n}}^k M$ of M .

2.3. Lemma. *If M is a (U_λ, K) -module then the generalised h -eigenspaces of \hat{M} are finite dimensional. For any integer m the sum of generalised h -eigenspaces with eigenvalue greater than m is finite dimensional. Furthermore, \hat{M} is the direct product of its generalised h -eigenspaces.*

Proof. As the space $M/\bar{\mathfrak{n}}M$ is finite dimensional, it gives rise to a finite set S of generalised h -eigenvalues. The elements in $\bar{\mathfrak{n}}^k M/\bar{\mathfrak{n}}^{k+1} M$ can be obtained by multiplying $M/\bar{\mathfrak{n}}M$ by $\bar{\mathfrak{n}}^k$, and so each eigenvalue of h on $\bar{\mathfrak{n}}^k M/\bar{\mathfrak{n}}^{k+1} M$ is a sum of an element of the set S with an integer less than or equal to $-k$. This gives our conclusion. \square

2.4. Proposition. *If M is a (U_λ, K) -module, then the (U_λ, N) -module $J(M)$ is naturally isomorphic to the direct sum of the generalised h -eigenspaces in \hat{M} .*

Proof. To prove the proposition, we must show that a vector in \hat{M} is \mathfrak{n} -finite if and only if it is a sum of generalised h -eigenvectors. Clearly any \mathfrak{n} -finite vector is such a sum since it lies in a finite-dimensional h -invariant subspace of \hat{M} . Conversely, by the previous lemma, the generalised h -eigenvalues of \hat{M} are bounded above, and so any sum of generalised h -eigenvectors is \mathfrak{n} -finite. \square

For $\alpha \in \mathbb{C}$, let \hat{M}_α denote the α -generalised eigenspace of h acting on \hat{M} . We define an increasing \mathbb{C} -filtration on \hat{M} by the formula

$$(2.2) \quad F_\alpha(\hat{M}) = \prod_{\beta \leq \alpha} \hat{M}_\beta.$$

Here and in what follows, for $\alpha, \beta \in \mathbb{C}$, we write $\beta \leq \alpha$ to mean $\alpha - \beta$ is a non-negative integer. Pulling back this filtration via the injection $M \rightarrow \hat{M}$, we obtain a filtration on M . The induced map on associated graded modules

$$(2.3) \quad \mathrm{Gr}_\bullet^F M \rightarrow \mathrm{Gr}_\bullet^F \hat{M}$$

is an isomorphism. Indeed the proof of Lemma 2.3 shows that the filtration just constructed is cofinal with the $\bar{\mathfrak{n}}$ -adic filtration on M , and thus that \hat{M} is the completion of M with respect to this filtration.

Similarly, we grade $\bar{\mathfrak{n}}$, and hence its enveloping algebra $U(\bar{\mathfrak{n}})$, according to the eigenvalues of the adjoint action of h . Let $U(\bar{\mathfrak{n}})_\beta$ denote the graded piece on which h acts with eigenvalue β . Note that for $U(\bar{\mathfrak{n}})_\beta$ to be non-zero, the eigenvalue β must be a non-positive integer. We define a filtration on $U(\bar{\mathfrak{n}})$ by the formula

$$(2.4) \quad F_{-k}(U(\bar{\mathfrak{n}})) = \bigoplus_{\beta \leq -k} U(\bar{\mathfrak{n}})_\beta.$$

The $U(\bar{\mathfrak{n}})$ -module structure on M is compatible with the filtrations on $U(\bar{\mathfrak{n}})$ and M .

2.5. Lemma. *Let α be a complex number.*

- (i) *For any $k \geq 0$, the quotient $F_{\alpha-k}(M)/(F_{-k}(U(\bar{\mathfrak{n}}))F_\alpha(M))$ is finite-dimensional.*
- (ii) *If $\alpha \in \mathbb{C}$ is sufficiently small, then for any integer $k \geq 0$, there is an equality $F_{\alpha-k}(M) = F_{-k}(U(\bar{\mathfrak{n}}))F_\alpha(M)$ (and so the quotient considered in (i) actually vanishes).*

Proof. The Lemma of Osborne guarantees that M is finitely generated over $U(\bar{\mathfrak{n}})$. Also, the filtrations constructed on each of $U(\bar{\mathfrak{n}})$ and M are cofinal with the $\bar{\mathfrak{n}}$ -adic filtrations. Thus the lemma follows from an easy variant of the Artin-Rees Lemma (applied to the filtrations F_\bullet rather than the $\bar{\mathfrak{n}}$ -adic filtrations). \square

3. NEARBY CYCLES, V -FILTRATION, AND THE GEOMETRIC JACQUET FUNCTOR

In this section, we recall the formalism of the V -filtration and the way it is used to define the nearby cycles functor on \mathcal{D} -modules. For a reference, see [K1].

As before, we write X for the flag manifold of \mathfrak{g} , and \mathcal{D}_λ for the sheaf of twisted differential operators with twist λ . We denote by t a coordinate on \mathbb{A}^1 , and consider the variety $\mathbb{A}^1 \times X$ and its subvariety $\{0\} \times X \simeq X$. We write $\tilde{\mathcal{D}}_\lambda$ for the sheaf of λ -twisted differential operators on $\mathbb{A}^1 \times X$. (Of course, the twisting only occurs along the X -factor.) The V -filtration on $\tilde{\mathcal{D}}_\lambda$ is the filtration where we declare that the variable t is of degree one, ∂_t is of degree -1 , and everything in \mathcal{D}_λ is of degree zero.

For a regular holonomic $\tilde{\mathcal{D}}_\lambda$ -module \mathcal{M} on $\mathbb{A}^1 \times X$, the V -filtration $V^\alpha \mathcal{M}$ is a decreasing filtration on \mathcal{M} , indexed by $\alpha \in \mathbb{C}$, compatible with the V -filtration on $\tilde{\mathcal{D}}_\lambda$. The V -filtration on \mathcal{M} is uniquely determined by the following properties:

there exist elements $u_1, \dots, u_k \in \mathcal{M}$ and $\mu_1, \dots, \mu_k \in \mathbb{C}$ such that

$$(3.1) \quad V^\alpha(\mathcal{M}) = \sum_{\alpha \leq m_i + \mu_i} V^{m_i}(\tilde{\mathcal{D}}_\lambda)u_i,$$

and

$$(3.2) \quad \mathrm{Gr}_V^\alpha \mathcal{M} \text{ is a generalised eigenspace of } t\partial_t \text{ of eigenvalue } \alpha.$$

Here for $\alpha \in \mathbb{C}$, we have written Gr_V^α for the quotient $V^\alpha(\mathcal{M})/V^{>\alpha}(\mathcal{M})$.

The nearby cycles of \mathcal{M} along $\{0\} \times X \simeq X$ are given by

$$(3.3) \quad \psi(\mathcal{M}) = \mathrm{Gr}_V^0 \mathcal{M} \oplus \bigoplus_{\substack{\alpha \in \mathbb{C}/\mathbb{Z} \\ \alpha \notin \mathbb{Z}}} \mathrm{Gr}_V^\alpha \mathcal{M}.$$

We are now ready to define the geometric Jacquet functor. Recall from the first section that via the cocharacter $\nu : \mathbb{G}_m \rightarrow G$, we have defined a \mathbb{G}_m -action on X . This

gives us the diagram

$$(3.4) \quad X \xleftarrow{a} \mathbb{G}_m \times X \xrightarrow{j} \mathbb{A}^1 \times X \xleftarrow{i} \{0\} \times X \xrightarrow{\sim} X$$

where a is the action map, and the maps j and i are the obvious inclusions. We define the geometric Jacquet functor Ψ by the formula

$$(3.5) \quad \Psi(\mathcal{M}) = \psi(j_* a^* \mathcal{M}), \quad \text{for a } (\mathcal{D}_\lambda, K)\text{-module } \mathcal{M}.$$

4. PROOF OF THE MAIN THEOREM

In this section, we give a proof of Theorem 1.1. To begin with, suppose that \mathcal{M} is a (\mathcal{D}_λ, K) -module arising as the localisation of a (U_λ, K) -module M , so that $\mathcal{M} = \mathcal{D}_\lambda \otimes_{U_\lambda} M$. Our aim is to explicitly write down the V -filtration on $j_* a^* \mathcal{M}$ on the level of its global sections \tilde{M} . We will make use of the diagram

$$(4.1) \quad \mathbb{G}_m \times X \xrightarrow{\tilde{a}} \mathbb{G}_m \times X \xrightarrow{p} X,$$

where p is the projection, and the map \tilde{a} is given by $\tilde{a}(g, x) = (g, gx)$ so that $a = \tilde{a} \circ p$.

To describe \tilde{M} as a vector space, we use the coordinates in the middle copy of $\mathbb{G}_m \times X$ in diagram (4.1), that is we take the global sections of $p^* \mathcal{M}$. This gives us

$$(4.2) \quad \tilde{M} = \Gamma(\mathbb{A}^1 \times X, j_* a^* \mathcal{M}) = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} M.$$

To describe the action of

$$(4.3) \quad \tilde{U}_\lambda = \Gamma(\mathbb{G}_m \times X, \tilde{\mathcal{D}}_\lambda) = \mathbb{C}[t, t^{-1}, t\partial_t] \otimes_{\mathbb{C}} U_\lambda,$$

we use the coordinates in the first copy of $\mathbb{G}_m \times X$ in diagram (4.1). It suffices to write down the action of the global vector field $t\partial_t$ and the global vector fields given by elements of \mathfrak{g} . To this end, let us recall some of our previous notation: we have the cocharacter $\nu : \mathbb{G}_m \rightarrow G$, and we write h for the semisimple element in \mathfrak{g} given by $d\nu(t\partial_t)$. First, under the map \tilde{a} the global vector field $(t\partial_t, 0)$ is mapped to the global vector field $(t\partial_t, h)$. Hence,

$$(4.4) \quad t\partial_t \cdot (f(t)m) = (t\partial_t f(t))m + f(t)(hm).$$

Second, under the map \tilde{a} the global vector field $(0, v)$ given by an element $v \in \mathfrak{g}$ is mapped to the global vector field $(0, \text{Ad}_{\nu^{-1}(t)} v)$. Hence,

$$(4.5) \quad v \cdot (f(t)m) = f(t)((\text{Ad}_{\nu^{-1}(t)} v)m).$$

In particular, if v is in the β root space \mathfrak{g}_β , then

$$(4.6) \quad v \cdot m = t^{-\langle \nu, \beta \rangle} vm.$$

(The inverse of the adjoint action arises for the following reason. For a vector $v \in \mathfrak{g}$ thought of as a vector field on X , the bracket $[h, v]$ may be calculated by pulling back v along the integral curves given by the action of $\nu(t)$. Thus the derivative of the pushforward of v under the action of $\nu(t)$ is the negative of $[h, v]$.)

To write down the V -filtration on \tilde{M} , let us recall the discussion of Section 2. There we constructed an increasing \mathbb{C} -filtration $F_\bullet(M)$ on M . We now argue that the V -filtration on \tilde{M} may be written in terms of this filtration as follows

$$(4.7) \quad V^\alpha(\tilde{M}) = \bigoplus_{k \in \mathbb{Z}} t^k F_{-\alpha+k}(M) \subset \bigoplus_{k \in \mathbb{Z}} t^k M = \tilde{M}.$$

To show that this is the V -filtration we must verify conditions (3.1) and (3.2). The fact that (3.2) holds is clear from the construction (taking into account equation 4.4). To show that (3.1) holds, we show that

$$(4.8) \quad V^{\alpha+m}(\tilde{M}) = V^m(\tilde{U}_\lambda) V^\alpha(\tilde{M}), \quad \text{for all } \alpha \text{ and } m,$$

and that

$$(4.9) \quad \text{each } V^\alpha(\tilde{M}) \text{ is finitely generated over } V^0(\tilde{U}_\lambda).$$

The first statement (4.8) is clear since multiplication by t clearly induces an isomorphism between $V^\alpha(\tilde{M})$ and $V^{\alpha+1}(\tilde{M})$. Therefore it remains to argue (4.9). We will in fact show that $V^\alpha(\tilde{M})$ is finitely generated over $\mathbb{C}[t] \otimes_{\mathbb{C}} U(\bar{\mathfrak{n}})$. By Lemmas 2.3 and 2.5, we have

$$(4.10) \quad F_{-\alpha+k}(M) = M, \text{ for sufficiently large } k,$$

$$(4.11) \quad F_{-\alpha-k}(M)/(F_{-k}(U(\bar{\mathfrak{n}}))F_\alpha(M)) \text{ is finite-dimensional, for all } k \geq 0,$$

and

$$(4.12) \quad F_{\alpha-k-l}(M) = F_{-l}(U(\bar{\mathfrak{n}}))F_{\alpha-k}(M), \text{ for sufficiently large } k \text{ and all } l \geq 0.$$

These three statements together (taking into account equation 4.6) imply that $V^\alpha(\tilde{M})$ is finitely generated over $\mathbb{C}[t] \otimes_{\mathbb{C}} U(\bar{\mathfrak{n}})$, and hence that it is finitely generated over $V^0(\tilde{U}_\lambda) = \mathbb{C}[t, t\partial_t] \otimes_{\mathbb{C}} U_\lambda$.

Now by Section 3, we have

$$(4.13) \quad \Psi(M) = \text{direct sum of generalised } h\text{-eigenspaces in } \hat{M}.$$

Part (i) of Theorem 1.1 thus follows from Proposition 2.4.

We turn to proving part (ii) of the theorem. Thus we suppose that \mathcal{M} is an arbitrary (\mathcal{D}_λ, K) -module. If we write $M = \Gamma(\mathcal{M})$, then there is a canonical map $\mathcal{D}_\lambda \otimes_{U_\lambda} M \rightarrow \mathcal{M}$, whose kernel and cokernel are both taken to zero by the exact functor Γ . Taking into account the fact that the passage to nearby cycles is exact, as well as part (i) of the theorem, we see that part (ii) follows from the next statement:

$$(4.14) \quad \text{if } \Gamma(\mathcal{M}) = 0, \text{ then } \Gamma(\Psi(\mathcal{M})) = 0.$$

To prove (4.14), we will argue as follows. First, it suffices to prove it in the case when \mathcal{M} is irreducible. Then, we will appeal to the results of [K2], §8. The final remark of [K2], p. 103, shows that we may find a Weyl group element w so that $w(\lambda)$ is anti-dominant, and satisfies condition (8.3.2). Applying the intertwining functor I_w induces an equivalence of categories between the category of (\mathcal{D}_λ, K) -modules and the category of $(\mathcal{D}_{w(\lambda)}, K)$ -modules, which is easily verified to be compatible with the functor Ψ on each of these categories. Thus in verifying 4.14 we may replace λ by $w(\lambda)$. The

remark preceding the statement of [K2], Prop. 8.2.1 shows that we may also replace G by its simply-connected cover, and thus assume that the sheaf $\mathcal{O}(\rho)$ is a G -equivariant line-bundle on X . Proposition 8.2.1 and Theorem 8.3.1 of [K2] now show there exists a simple root α such that $\check{\alpha}(\lambda) = 0$ and $\mathcal{M} = \mathcal{O}(-\rho) \otimes p^* \mathcal{M}'$, where $p : X \rightarrow X_\alpha$ is the projection from the full flag manifold X to the partial flag manifold X_α of parabolics of the type associated to the simple root α , and \mathcal{M}' is a $\mathcal{D}_{\lambda+\rho}$ -module on X_α . The functor Ψ has its analogue Ψ_α defined on $\mathcal{D}_{\lambda+\rho}$ -modules on X_α , compatible with Ψ . Thus we conclude

$$(4.15) \quad \begin{aligned} \Gamma(\Psi(\mathcal{M})) &= \Gamma(\Psi(\mathcal{O}(-\rho) \otimes p^* \mathcal{M}')) = \Gamma(\mathcal{O}(-\rho) \otimes \Psi(p^* \mathcal{M}')) \\ &= \Gamma(\mathcal{O}(-\rho) \otimes p^* \Psi_\alpha(\mathcal{M}')) = \Gamma(p_*(\mathcal{O}(-\rho) \otimes p^* \Psi_\alpha(\mathcal{M}'))) = 0; \end{aligned}$$

here the second equality follows from the evident compatibility of the functor Ψ with twisting by G -equivariant line-bundles, the third equality follows from the compatibility of p^* with Ψ and Ψ_α , and the last equality holds because, by [K2], Prop. 8.2.1, we see that $p_*(\mathcal{O}(-\rho) \otimes p^*(\mathcal{M}'))$ is zero for any \mathcal{D}_λ -module \mathcal{M}' on X_α , when $\check{\alpha}(\lambda) = 0$.

5. A GEOMETRIC INTERPRETATION

As mentioned in the introduction, the arguments which go into the proof of our main theorem are primarily algebraic. In this section, we informally describe the geometry those arguments formalise and also discuss explicitly the example $G_{\mathbb{R}} = \mathrm{SL}_2(\mathbb{R})$. The discussion is in the language of λ -twisted perverse sheaves rather than holonomic \mathcal{D}_λ -modules.

To review our notation, $\mathfrak{p}_{\mathbb{R}}$ is a minimal parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$ with complexification \mathfrak{p} ; \mathfrak{n} is the nilpotent radical of \mathfrak{p} ; P is the parabolic subgroup with Lie algebra \mathfrak{p} ; \bar{P} is the parabolic subgroup opposite to P ; L is the Levi subgroup $P \cap \bar{P}$; and X is the flag manifold of \mathfrak{g} .

The basic ingredient in defining the geometric Jacquet functor is the action

$$(5.1) \quad a : \mathbb{G}_m \times X \rightarrow X$$

defined as the composition of the action of G on X with a cocharacter $\nu : \mathbb{G}_m \rightarrow G$. Recall that we choose ν such that when paired with the roots of G it is positive precisely on the roots in \mathfrak{n} .

The action a defines two Morse stratifications of X , one by ascending manifolds, one by descending manifolds, which have the same critical manifolds. The ascending manifolds are the P -orbits, the descending manifolds are the \bar{P} -orbits, and the critical manifolds are the L -orbits. Indeed, each connected component X^α of the fixed points of a is a single L -orbit, the P -orbit X_P^α through X^α satisfies

$$(5.2) \quad X_P^\alpha = \{x \in X \mid \lim_{z \rightarrow 0} a(z) \cdot x \in X^\alpha\},$$

and the \bar{P} -orbit $X_{\bar{P}}^\alpha$ through X^α satisfies

$$(5.3) \quad X_{\bar{P}}^\alpha = \{x \in X \mid \lim_{z \rightarrow \infty} a(z) \cdot x \in X^\alpha\}.$$

Given a λ -twisted K -equivariant perverse sheaf \mathcal{M} on X we think of its geometric Jacquet module $\Psi(\mathcal{M})$ as the result of "flowing" \mathcal{M} along the trajectories of the action

a and "passing to the limit". The limit $\Psi(\mathcal{M})$ is constructible with respect to the stratification by the ascending manifolds $\{X_P^\alpha\}$, and an ascending manifold X_P^α is in the support of $\Psi(\mathcal{M})$ exactly when the corresponding descending manifold X_P^α intersects the support of \mathcal{M} . The restriction of $\Psi(\mathcal{M})$ to a critical manifold X^α may be calculated as the integral of \mathcal{M} over the fibers of the corresponding descending manifold X_P^α .

To see this in a concrete example, consider the real group $\mathrm{SL}_2(\mathbb{R})$, with maximal compact subgroup $\mathrm{SO}_2(\mathbb{R})$, and the corresponding Harish-Chandra pair $(\mathfrak{sl}_2(\mathbb{C}), \mathrm{SO}_2(\mathbb{C}))$. We may identify the flag manifold of $\mathfrak{sl}_2(\mathbb{C})$ with the complex projective line $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ so that (i) the $\mathrm{SO}_2(\mathbb{C})$ -orbits are the points i and $-i$, and their complement $\mathbb{CP}^1 \setminus \{i, -i\}$, and (ii) the point 0 represents a Borel subalgebra $\mathfrak{b} \subset \mathfrak{sl}_2(\mathbb{C})$ which is the complexification of a minimal parabolic subalgebra of $\mathfrak{p}_{\mathbb{R}} \subset \mathfrak{sl}_2(\mathbb{R})$.

Fix a character $\lambda : \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \rightarrow \mathbb{C}$. Let δ_0 be the irreducible λ -twisted perverse sheaf on \mathbb{CP}^1 supported at the point 0, and let \mathcal{N} be the middle-extension to \mathbb{CP}^1 of the irreducible λ -twisted local system on $\mathbb{CP}^1 \setminus \{0\}$. For λ non-integral, there are two irreducible λ -twisted $\mathrm{SO}_2(\mathbb{C})$ -equivariant perverse sheaves on \mathbb{CP}^1 : the middle-extensions \mathcal{M}_1 and \mathcal{M}_{-1} of the irreducible trivial and twisted $\mathrm{SO}_2(\mathbb{C})$ -equivariant local systems on the complement $\mathbb{CP}^1 \setminus \{i, -i\}$. For λ integral, there are in addition two other irreducible λ -twisted $\mathrm{SO}_2(\mathbb{C})$ -equivariant perverse sheaves on \mathbb{CP}^1 : the irreducible $\mathrm{SO}_2(\mathbb{C})$ -equivariant perverse sheaves δ_i and δ_{-i} supported at the points i and $-i$.

Choose the cocharacter $\nu : \mathbb{C}^\times \rightarrow \mathrm{SL}_2(\mathbb{C})$ so that the induced action $a : \mathbb{C}^\times \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ is the standard multiplication $a(z) \cdot p = z^{-2}p$. Note that in the limit $z \rightarrow \infty$, the points i and $-i$ flow to 0, and this completely describes what happens to the $\mathrm{SO}_2(\mathbb{C})$ -orbits. With this picture in mind, we calculate the geometric Jacquet module of the irreducible λ -twisted $\mathrm{SO}_2(\mathbb{C})$ -equivariant perverse sheaves on \mathbb{CP}^1 . For λ integral, we have

$$\Psi(\delta_i) = \Psi(\delta_{-i}) = \delta_0 \quad \Psi(\mathcal{M}_1) = \mathcal{N} \quad \Psi(\mathcal{M}_{-1}) = \mathcal{L}$$

where \mathcal{L} is the unique non-semisimple self-dual perverse sheaf with three-step filtration $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}$ satisfying

$$\mathcal{L}_0 = \delta_0 \quad \mathcal{L}_1/\mathcal{L}_0 = \mathcal{N} \quad \mathcal{L}/\mathcal{L}_1 = \delta_0.$$

For λ non-integral, we have

$$\Psi(\mathcal{M}_1) = \Psi(\mathcal{M}_{-1}) = \mathcal{N} \oplus \delta_0 \oplus \delta_0.$$

To help explain why the result for \mathcal{M}_1 and λ integral has different simple constituents from the other perverse sheaves supported on the open orbit, note that only for \mathcal{M}_1 and λ integral is the characteristic variety of such a perverse sheaf the zero section alone.

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